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On a conjecture by Haipeng Qu

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Abstract. In this note, we prove that $D_8 \times C_2^{n-3}$ is the non-elementary abelian 2-group of order 2^n, $n \geq 3$, whose number of subgroups of possible orders is maximal. This solves a conjecture by Haipeng Qu. A formula for counting the subgroups of an (almost) extraspecial 2-group is also presented.

1 Introduction

Let G be a finite p-group of order p^n. For $k = 0, 1, \ldots, n$, we denote by $s_k(G)$ the number of subgroups of order p^k of G. The starting point for our discussion is given by [7, Theorem 1.4] which proves that if p is odd and G is non-elementary abelian, then

$$s_k(G) \leq s_k(M_p \times C_p^{n-3}) \quad \text{for all } 0 \leq k \leq n, \tag{1.1}$$

where

$$M_p = \langle a, b \mid a^p = b^p = c^p = 1,\ [a,b] = c,\ [c,a] = [c,b] = 1 \rangle$$

is the non-abelian group of order p^3 and exponent p. Moreover, in [7] it is conjectured that the inequalities in (1.1) also hold for $p = 2$. Note that in this case there is no direct analogue of M_p, as a group of exponent 2 is abelian. But the dihedral group of order 8 is close, as it is at least generated by elements of order 2.

In the current note we prove the inequalities in (1.1) for $p = 2$ by replacing M_p with D_8. This completes the work of Haipeng Qu. We also give a formula for the number of subgroups of an (almost) extraspecial 2-group depending on the number of elementary abelian subgroups of possible orders. Our main result is the following.

Theorem 1.1. *Let G be a finite non-elementary abelian 2-group of order 2^n, where $n \geq 3$. Then*

$$s_k(G) \leq s_k(D_8 \times C_2^{n-3}) \quad \textit{for all } 0 \leq k \leq n. \tag{1.2}$$

Throughout this paper, we will use $A * B$ to denote the (amalgamated) central product of the groups A and B having isomorphic centres, and X^{*r} to denote the central product of r copies of the group X. Also, we will denote by $\binom{n}{k}_p$ the number of subgroups of order p^k in an elementary abelian p-group of order p^n. Other notation is standard and can be found in books such as [1, 4].

We recall several basic definitions and results on 2-groups that will be useful to us. A finite 2-group is called:

- *extraspecial* if $Z(G) = G' = \Phi(G)$ has order 2,
- *almost extraspecial* if $G' = \Phi(G)$ has order 2 and $Z(G) \cong C_4$,
- *generalized extraspecial* if $G' = \Phi(G)$ has order 2 and $G' \subseteq Z(G)$.

The structure of these groups is well known (see, for example, [2, Theorem 2.3] and [8, Lemma 3.2]):

Lemma 1.2. *Let G be a finite 2-group.*

(a) *If G is extraspecial, then $|G| = 2^{2r+1}$ for some positive integer r and either $G \cong D_8^{*r}$ or $G \cong Q_8 * D_8^{*(r-1)}$.*

(b) *If G is almost extraspecial, then $|G| = 2^{2r+2}$ for some positive integer r and $G \cong D_8^{*r} * C_4$.*

(c) *If G is generalized extraspecial, then either $G \cong E \times A$ or $G \cong (E * C_4) \times A$, where E is an extraspecial 2-group and A is an elementary abelian 2-group.*

The key result of [7] is Theorem 1.3. For $p = 2$, it states the following.

Lemma 1.3. *Let G be a finite 2-group of order 2^n and let M be a normal subgroup of order 2 of G. Then*

$$s_k(G) \le s_k(G/M \times C_2) \quad \text{for all } 0 \le k \le n.$$

We also present a result that follows from [3, Corollary 2].

Lemma 1.4. *Let G be a finite non-elementary abelian 2-group of order 2^n, $n \ge 3$, and let $L_1(G)$ be the set of cyclic subgroups of G. Then*

$$|L_1(G)| \le 7 \cdot 2^{n-3} = |L_1(D_8 \times C_2^{n-3})|.$$

2 Proofs of the main results

First of all, we recall the well-known Goursat's lemma (see e.g. [9, (4.19)]) that will be used intensively in our proofs.

Theorem 2.1. *Let A and B be two finite groups. Then every subgroup H of the direct product $A \times B$ is completely determined by a quintuple $(A_1, A_2, B_1, B_2, \varphi)$, where $A_1 \trianglelefteq A_2 \leq A$, $B_1 \trianglelefteq B_2 \leq B$ and $\varphi : A_2/A_1 \to B_2/B_1$ is an isomorphism, more precisely*

$$H = \{(a, b) \in A_2 \times B_2 \mid \varphi(aA_1) = bB_1\}.$$

Moreover, we have $|H| = |A_1||B_2| = |A_2||B_1|$.

Next we remark that Lemma 1.3 can easily be generalized in the following way.

Lemma 2.2. *Let G be a finite 2-group of order 2^n and let M be a normal subgroup of order 2^r of G. Then*

$$s_k(G) \leq s_k(G/M \times C_2^r) \quad \textit{for all } 0 \leq k \leq n.$$

Proof. Take a chief series of G containing M and use induction on r. □

The following lemma shows that the inequalities in (1.2) hold for finite abelian 2-groups.

Lemma 2.3. *Let G be a finite abelian 2-group of order 2^n, $n \geq 3$. If G is non-elementary abelian, then*

$$s_k(G) \leq s_k(D_8 \times C_2^{n-3}) \quad \textit{for all } 0 \leq k \leq n.$$

Proof. Since G is not elementary abelian, we infer that it has a subgroup M of order 2^{n-2} such that $G/M \cong C_4$. Then Lemma 2.2 leads to

$$s_k(G) \leq s_k(C_4 \times C_2^{n-2}) \quad \text{for all } 0 \leq k \leq n,$$

and so it suffices to prove that

$$s_k(G_1 \times C_2^{n-3}) \leq s_k(D_8 \times C_2^{n-3}) \quad \text{for all } 0 \leq k \leq n, \tag{2.1}$$

where $G_1 = C_4 \times C_2$. By Theorem 2.1, we know that a subgroup of order 2^k of $G_1 \times C_2^{n-3}$ is completely determined by a quintuple $(A_1, A_2, B_1, B_2, \varphi)$, where $A_1 \trianglelefteq A_2 \leq G_1$, $B_1 \trianglelefteq B_2 \leq C_2^{n-3}$, $\varphi : A_2/A_1 \to B_2/B_1$ is an isomorphism, and $|A_2||B_1| = 2^k$. Note that φ can be chosen in a unique way for the elementary abelian sections of orders 1 and 2 of G_1, and in $6 = |\mathrm{Aut}(C_2^2)|$ ways for the elementary abelian sections of order 4 of G_1. We distinguish the following cases:

Case (a): $A_2 = 1$. Then $A_1 = 1$ and $B_1 = B_2$ is one of the $\binom{n-3}{k}_2$ subgroups of order 2^k of C_2^{n-3}. Clearly, these determine $\binom{n-3}{k}_2$ distinct subgroups of the group $G_1 \times C_2^{n-3}$.

Case (b): A_2 is one of the three subgroups of order 2 of G_1. Then B_1 is of order 2^{k-1} and can be chosen in $\binom{n-3}{k-1}_2$ ways. If $A_1 = A_2$, then we have $B_2 = B_1$, while if $A_1 = 1$, then B_2 is one of the $2^{n-k-2} - 1$ subgroups of order 2^k of C_2^{n-3} containing B_1. So, in this case we have

$$3 \cdot \binom{n-3}{k-1}_2 + 3 \cdot (2^{n-k-2} - 1)\binom{n-3}{k-1}_2 = 3 \cdot 2^{n-k-2}\binom{n-3}{k-1}_2$$

distinct subgroups of $G_1 \times C_2^{n-3}$.

Case (c): A_2 is one of the two cyclic subgroups of order 4 of G_1. Then B_1 is of order 2^{k-2} and can be chosen in $\binom{n-3}{k-2}_2$ ways. If $A_1 = A_2$, then we have $B_2 = B_1$, while if $|A_1| = 2$, then B_2 is one of the $2^{n-k-1} - 1$ subgroups of order 2^{k-1} of C_2^{n-3} containing B_1. So, in this case we have

$$2 \cdot \binom{n-3}{k-2}_2 + 2 \cdot (2^{n-k-1} - 1)\binom{n-3}{k-1}_2 = 2^{n-k}\binom{n-3}{k-2}_2$$

distinct subgroups of $G_1 \times C_2^{n-3}$.

Case (d): A_2 is the unique subgroup isomorphic to C_2^2 of G_1. Again, B_1 is of order 2^{k-2} and can be chosen in $\binom{n-3}{k-2}_2$ ways. If $A_1 = A_2$, then we have $B_2 = B_1$; if $|A_1| = 2$, then B_2 is one of the $2^{n-k-1} - 1$ subgroups of order 2^{k-1} of C_2^{n-3} containing B_1; if $A_1 = 1$, then B_2 is one of the $\binom{n-k-1}{2}_2$ subgroups of order 2^k of C_2^{n-3} containing B_1. One obtains

$$\binom{n-3}{k-2}_2 + 3(2^{n-k-1}-1)\binom{n-3}{k-2}_2 + 6\binom{n-k-1}{2}_2\binom{n-3}{k-2}_2 = 2^{2n-2k-2}\binom{n-3}{k-2}_2$$

distinct subgroups of $G_1 \times C_2^{n-3}$.

Case (e): $A_2 = G_1$. In this case B_1 is of order 2^{k-3} and can be chosen in $\binom{n-3}{k-3}_2$ ways. If $A_1 = A_2$, then $B_2 = B_1$; if $|A_1| = 4$, then B_2 is one of the $2^{n-k} - 1$ subgroups of order 2^{k-2} of C_2^{n-3} containing B_1; if A_1 is the unique subgroup of order 2 of G_1 such that $A_2/A_1 \cong C_2^2$, then B_2 is one of the $\binom{n-k}{2}_2$ subgroups of order 2^{k-1} of C_2^{n-3} containing B_1. One obtains

$$\binom{n-3}{k-3}_2 + 3(2^{n-k} - 1)\binom{n-3}{k-3}_2 + 6\binom{n-k}{2}_2\binom{n-3}{k-3}_2 = 2^{2n-2k}\binom{n-3}{k-3}_2$$

distinct subgroups of $G_1 \times C_2^{n-3}$.

By summing all the above quantities, we get

$$s_k(G_1 \times C_2^{n-3}) = \binom{n-3}{k}_2 + 3 \cdot 2^{n-k-2}\binom{n-3}{k-1}_2 + 2^{n-k}(2^{n-k-2} + 1)\binom{n-3}{k-2}_2 + 2^{2n-2k}\binom{n-3}{k-3}_2.$$

A similar computation leads to

$$s_k(D_8 \times C_2^{n-3}) = \binom{n-3}{k}_2 + 5 \cdot 2^{n-k-2}\binom{n-3}{k-1}_2 + 2^{n-k-1}(2^{n-k}+1)\binom{n-3}{k-2}_2 + 2^{2n-2k}\binom{n-3}{k-3}_2.$$

It is now clear that the inequalities in (2.1) are true, completing the proof. □

In what follows we will focus on describing the subgroup lattice of an (almost) extraspecial 2-group G. This can be easily made by using [2, Lemma 2.6].

Lemma 2.4. *If G is an (almost) extraspecial 2-group, then $L(G)$ consists of:*

(a) *the trivial subgroup.*

(b) *all subgroups containing $\Phi(G)$; moreover, these are the normal non-trivial subgroups of G.*

(c) *all complements of $\Phi(G)$ in the elementary abelian subgroups of order ≥ 4 containing $\Phi(G)$; moreover,*
- *$\Phi(G)$ has 2^i complements in an elementary abelian subgroup of order 2^{i+1} containing $\Phi(G)$,*
- *two non-normal subgroups H and K of G are conjugate if and only if $H\Phi(G) = K\Phi(G)$,*
- *given two non-normal subgroups H and K of G, if $H^x \subseteq K$ for some $x \in [G/N_G(H)]$, then x is the unique element of $[G/N_G(H)]$ with this property.*

It is well known that the order of maximal elementary abelian subgroups of an extraspecial 2-group of order 2^{2r+1} or of an almost extraspecial 2-group of order 2^{2r+2} is 2^{r+1}. Let us denote by $e_i(G)$ the number of elementary abelian subgroups of order 2^i containing $\Phi(G)$, $i = 2, 3, \ldots, r+1$. Under this notation, a formula for the number of subgroups of G can be inferred from Lemma 2.4.

Corollary 2.5. *If G is an extraspecial 2-group of order 2^{2r+1}, then*

$$|L(G)| = 1 + \sum_{i=0}^{2r} \binom{2r}{i}_2 + \sum_{i=1}^{r} e_{i+1}(G)2^i, \tag{2.2}$$

while if G is an almost extraspecial 2-group of order 2^{2r+2}, then

$$|L(G)| = 1 + \sum_{i=0}^{2r+1} \binom{2r+1}{i}_2 + \sum_{i=1}^{r} e_{i+1}(G)2^i. \tag{2.3}$$

In the particular cases $r = 2$ and $r = 1$, equalities (2.2) and (2.3), respectively, lead to some known results (see e.g. [5, Section 3.3] and [6, Example 4.5]).

Examples. (a) For $G = D_8 * D_8$ we obtain $e_2(G) = 9$ and $e_3(G) = 6$, implying that

$$|L(G)| = 1 + \sum_{i=0}^{4} \binom{4}{i}_2 + 2 \cdot 9 + 4 \cdot 6 = 110.$$

(b) For $G = Q_8 * D_8$ we obtain $e_2(G) = 5$ and $e_3(G) = 0$, implying that

$$|L(G)| = 1 + \sum_{i=0}^{4} \binom{4}{i}_2 + 2 \cdot 5 = 78.$$

(c) For $G = D_8 * C_4$ we obtain $e_2(G) = 3$, and so

$$|L(G)| = 1 + \sum_{i=0}^{3} \binom{3}{i}_2 + 2 \cdot 3 = 23.$$

Remark. As we have seen above, the subgroup structure of an (almost) extraspecial 2-group G depends on its elementary abelian subgroups containing the subgroup $\Phi(G) = \langle x \rangle$. We remark that these are the totally singular subspaces of $G/\Phi(G)$ with respect to the quadratic form $q : G/\Phi(G) \to \mathbb{F}_2$, where $q(\bar{v})$ is the element $a \in \mathbb{F}_2$ such that $v^2 = x^a$ for all $\bar{v} \in G/\Phi(G)$.

Next we will compare the subgroup lattices of an (almost) extraspecial 2-group G of order 2^n, $n \geq 3$, and of $D_8 \times C_2^{n-3}$. Since $\Phi(D_8 \times C_2^{n-3})$ is of order 2, it follows that $D_8 \times C_2^{n-3}/\Phi(D_8 \times C_2^{n-3})$ and $G/\Phi(G)$ have the same dimension over $\mathbb{F}_2$, namely $n-1$. Also, we note that the order of maximal elementary abelian subgroups of $D_8 \times C_2^{n-3}$ is 2^{n-1}, which is greater than or equal to the order of maximal elementary abelian subgroups of G. We infer that

$$|L(D_8 \times C_2^{n-3})| = 1 + \sum_{i=0}^{n-1} \binom{n-1}{i}_2 + \sum_{i=1}^{n-2} e_{i+1}(D_8 \times C_2^{n-3}) 2^i.$$

The following lemma will be crucial in our proof.

Lemma 2.6. *Under the above notation, we have $e_2(G) \leq e_2(D_8 \times C_2^{n-3})$.*

Proof. By Lemma 1.4, we know that

$$|L_1(G)| \leq |L_1(D_8 \times C_2^{n-3})|.$$

Let $c_i(G)$ be the number of cyclic subgroups of order 2^i of G. Since both G and $D_8 \times C_2^{n-3}$ are of exponent 4, the above inequality can be rewritten as

$$1 + c_2(G) + c_4(G) \leq 1 + c_2(D_8 \times C_2^{n-3}) + c_4(D_8 \times C_2^{n-3}).$$

On the other hand, we have

$$2^n = 1 + c_2(G) + 2c_4(G) = 1 + c_2(D_8 \times C_2^{n-3}) + 2c_4(D_8 \times C_2^{n-3})$$

and consequently

$$2^n - c_4(G) \le 2^n - c_4(D_8 \times C_2^{n-3}),$$

i.e.

$$c_4(D_8 \times C_2^{n-3}) \le c_4(G). \tag{2.4}$$

It is clear that the cyclic subgroups of order 4 of these groups contain the Frattini subgroup. Thus (2.4) leads to $e_2(G) \le e_2(D_8 \times C_2^{n-3})$, as desired. □

Since any elementary abelian subgroup of G containing $\Phi(G)$ is a direct sum of elementary abelian subgroups of order 4, from Lemma 2.6 we infer that

$$e_i(G) \le e_i(D_8 \times C_2^{n-3}) \quad \text{for all } i.$$

An immediate consequence of this fact is that the inequalities in (1.2) also hold for (almost) extraspecial 2-groups.

Corollary 2.7. *If G is an (almost) extraspecial 2-group of order 2^n, $n \ge 3$, then*

$$s_k(G) \le s_k(D_8 \times C_2^{n-3}) \quad \textit{for all } 0 \le k \le n.$$

A similar thing can be said about elementary abelian sections of the groups G and $D_8 \times C_2^{n-3}$.

Corollary 2.8. *Given a 2-group G, we denote by $\mathcal{S}_{(\alpha,\beta)}(G)$ the set of all elementary abelian sections $H_2/H_1 \cong C_2^\alpha$ of G with $|H_1| = 2^\beta$. If G is (almost) extraspecial of order 2^n, then*

$$|\mathcal{S}_{(\alpha,\beta)}(G)| \le |\mathcal{S}_{(\alpha,\beta)}(D_8 \times C_2^{n-3})| \quad \textit{for all } \alpha \textit{ and } \beta.$$

Proof. Write

$$\mathcal{S}_{(\alpha,\beta)}(G) = \mathcal{S}^1_{(\alpha,\beta)}(G) \cup \mathcal{S}^2_{(\alpha,\beta)}(G) \cup \mathcal{S}^3_{(\alpha,\beta)}(G) \cup \mathcal{S}^4_{(\alpha,\beta)}(G),$$

where

$$\mathcal{S}^1_{(\alpha,\beta)}(G) = \{H_2/H_1 \in \mathcal{S}_{(\alpha,\beta)}(G) \mid \Phi(G) \subseteq H_1\},$$
$$\mathcal{S}^2_{(\alpha,\beta)}(G) = \{H_2/H_1 \in \mathcal{S}_{(\alpha,\beta)}(G) \mid H_1 = 1\},$$
$$\mathcal{S}^3_{(\alpha,\beta)}(G) = \{H_2/H_1 \in \mathcal{S}_{(\alpha,\beta)}(G) \mid \Phi(G) \nsubseteq H_2,\ H_1 \ne 1\},$$
$$\mathcal{S}^4_{(\alpha,\beta)}(G) = \{H_2/H_1 \in \mathcal{S}_{(\alpha,\beta)}(G) \mid \Phi(G) \subseteq H_2,\ \Phi(G) \nsubseteq H_1,\ H_1 \ne 1\}.$$

Since $G/\Phi(G) \cong D_8 \times C_2^{n-3}/\Phi(D_8 \times C_2^{n-3})$, we have

$$|\mathcal{S}^1_{(\alpha,\beta)}(G)| = |\mathcal{S}^1_{(\alpha,\beta)}(D_8 \times C_2^{n-3})|. \tag{2.5}$$

Clearly,

$$\mathcal{S}^2_{(\alpha,\beta)}(G) = \mathcal{S}_{(\alpha,0)}(G)$$

is the number of elementary abelian subgroups of order 2^α of G, and so

$$|\mathcal{S}^2_{(\alpha,\beta)}(G)| = e_\alpha(G) + e_{\alpha+1}(G)2^\alpha,$$

implying that

$$|\mathcal{S}^2_{(\alpha,\beta)}(G)| \le |\mathcal{S}^2_{(\alpha,\beta)}(D_8 \times C_2^{n-3})|. \tag{2.6}$$

Furthermore, we observe that every section $H_2/H_1 \in \mathcal{S}^3_{(\alpha,\beta)}(G)$ determines a section $H_2\Phi(G)/H_1\Phi(G) \in \mathcal{S}^1_{(\alpha,\beta+1)}(G)$ with

$$H_2\Phi(G) \cong C_2^{\alpha+\beta+1} \quad \text{and} \quad H_1\Phi(G) \cong C_2^{\beta+1}.$$

Conversely, every section $A/B \in \mathcal{S}^1_{(\alpha,\beta+1)}(G)$ with

$$A \cong C_2^{\alpha+\beta+1} \quad \text{and} \quad B \cong C_2^{\beta+1}$$

determines 2^β sections $H_2/H_1 \in \mathcal{S}^3_{(\alpha,\beta)}(G)$. As there are $e_{\alpha+\beta+1}(G)\binom{\alpha+\beta+1}{\beta+1}_2$ such sections $A/B \in \mathcal{S}^1_{(\alpha,\beta+1)}(G)$, we infer that

$$|\mathcal{S}^3_{(\alpha,\beta)}(G)| = e_{\alpha+\beta+1}(G)\binom{\alpha+\beta+1}{\beta+1}_2 2^\beta \le |\mathcal{S}^3_{(\alpha,\beta)}(D_8 \times C_2^{n-3})|. \tag{2.7}$$

Let $H_2/H_1 \in \mathcal{S}^4_{(\alpha,\beta)}(G)$. Then $\Phi(H_2) \subseteq H_1$. On the other hand, we have

$$\Phi(H_2) \subseteq \Phi(G)$$

because H_2 is normal in G. Thus $\Phi(H_2) = 1$, i.e. H_2 is elementary abelian, and it can be chosen in $e_{\alpha+\beta}(G)$ ways. Also, H_1 is a complement of $\Phi(G)$ in one of the $\binom{\alpha+\beta}{\beta+1}_2$ subgroups of order $2^{\beta+1}$ of H_2, and it can be chosen in $\binom{\alpha+\beta}{\beta+1}_2 2^\beta$ ways. Hence

$$|\mathcal{S}^4_{(\alpha,\beta)}(G)| = e_{\alpha+\beta}(G)\binom{\alpha+\beta}{\beta+1}_2 2^\beta \le |\mathcal{S}^4_{(\alpha,\beta)}(D_8 \times C_2^{n-3})|. \tag{2.8}$$

Obviously, relations (2.5)–(2.8) lead to

$$|\mathcal{S}_{(\alpha,\beta)}(G)| \le |\mathcal{S}_{(\alpha,\beta)}(D_8 \times C_2^{n-3}),$$

as desired. □

We are now able to prove our main result.

Proof of Theorem 1.1. Let $|G'| = 2^m$. If $m = 0$, then G is abelian and the conclusion follows from Lemma 2.3. Assume that $m \geq 1$.

If G/G' is not elementary abelian, then

$$s_k(G) \leq s_k(G/G' \times C_2^m) \leq s_k(D_8 \times C_2^{n-3}) \quad \text{for all } 0 \leq k \leq n,$$

where the first inequality is obtained by Lemma 2.2, while the second one by Lemma 2.3.

If G/G' is elementary abelian, then $G' = \Phi(G)$. Let M be a normal subgroup of G such that $M \subseteq G'$ and $[G' : M] = 2$. Then

$$s_k(G) \leq s_k(G_1 \times C_2^{m-1}) \quad \text{for all } 0 \leq k \leq n,$$

where $G_1 = G/M$ satisfies the conditions

$$G_1' = \Phi(G_1), \quad |G_1'| = 2 \quad \text{and} \quad G_1' \subseteq Z(G_1),$$

i.e. it is a generalized extraspecial 2-group. Then either

$$G_1 \cong E \times A \quad \text{or} \quad G_1 \cong (E * C_4) \times A,$$

where E is an extraspecial 2-group and A is an elementary abelian 2-group, by Lemma 1.2. In other words, G_1 is a direct product of an (almost) extraspecial 2-group and an elementary abelian 2-group. So, it suffices to prove that if G_2 is an (almost) extraspecial 2-group of order 2^q, then

$$s_k(G_2 \times C_2^{n-q}) \leq s_k((D_8 \times C_2^{q-3}) \times C_2^{n-q}) \quad \text{for all } 0 \leq k \leq n. \tag{2.9}$$

Let A be one of the groups G_2 or $D_8 \times C_2^{q-3}$. From Theorem 2.1 it follows that a subgroup $H \leq A \times C_2^{n-q}$ of order 2^k is completely determined by a quintuple $(A_1, A_2, B_1, B_2, \varphi)$, where $A_1 \trianglelefteq A_2 \leq A$, $B_1 \trianglelefteq B_2 \leq C_2^{n-q}$, $\varphi : A_2/A_1 \to B_2/B_1$ is an isomorphism, and $|A_2||B_1| = 2^k$. By fixing the section B_2/B_1 of C_2^{n-q} and $\varphi \in \mathrm{Aut}(B_2/B_1)$, we infer that H depends only on the choice of the section $A_2/A_1 \in \mathcal{S}_{(\alpha,\beta)}(A)$, where $(\alpha, \beta) \in \{(i, j) \mid i = 0, 1, \dots, k,\ j = 0, 1, \dots, k-i\}$. So, to prove the inequalities in (2.9), it suffices to compare the numbers of elementary abelian sections A_2/A_1 of the two groups G_2 and $D_8 \times C_2^{q-3}$, where $|A_1|$ and $|A_2|$ are arbitrary. It is now clear that the conclusion follows from Corollary 2.8, completing the proof. □

Finally, we mention that a result similar to Lemma 1.4 can be obtained from Theorem 1.1.

Corollary 2.9. *Let G be a finite non-elementary abelian 2-group of order 2^n, $n \geq 3$. Then*

$$|L(G)| \leq |L(D_8 \times C_2^{n-3})|.$$

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